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Fourth-Kind Chebyshev Operational Tau Algorithm for Fractional Bagley-Torvik Equation

M. A. Taema [1,*], M. R. Zeen El-Deen [2], Y. H. Youssri [3]

[1]Faculty of Engineering, King Salman International University, El-Tur, Egypt

[2]Department of Mathematics , Faculty of Science , Suez University

[3]Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt

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ABSTRACT

In this work, a new numerical method for solving the Fractional Bagley-Torvik problem is established. The fundamental idea behind this novel approach is the clever incorporation of fourth-kind Chebyshev polynomials into the well-known operational tau technique.This study's main goal is to improve the accuracy and efficiency of the Fractional Bagley-Torvik equation solution. Managing non-homogeneous boundary conditions effectively is a crucial breakthrough that makes this possible. It is possible to transform these non-homogeneous situations into a more controllable and tractable homogenous form by using a methodical transformation process. This transformation phase enhances the numerical method's overall accuracy and efficiency while greatly streamlining the solution procedure. The study includes a number of carefully chosen numerical examples to confirm the effectiveness and usefulness of this suggested method. The accuracy and resilience of the Chebyshev polynomial-based operational tau approach in handling the intricacies of the Fractional Bagley-Torvik equation are demonstrated by these actual examples. By using these examples, the study hopes to demonstrate convincingly that this new approach provides a workable and efficient way to solve this difficult class of differential equations.

1. Introduction

Fractional differential equations (FDEs) have been studied in great detail in a variety of fields, including applied mathematics, finance, engineering, and other areas of applications [1, 2]. The new properties of these fractional differential and integral operators have led to a mass generalization of classical models to their fractional version in recent decades. Many fractional derivative operators, including the Caputo operator, Riemann-Liouville operator, Hadamard operator, and others, have been used extensively in scientific research [3, 4]. FDEs are used to model a lot of problems in a variety of fields, including the fractional Bagley-Torvik (B-T) equation [5]. At the same time, they have been studied both analytically and numerically [3], and numerous researchers have solved this equation numerically. Furthermore, this problem has been numerically solved by numerous experts. For instance, in [6], Abu Arqub and Maayah used an iterative reproducing kernel approach to solve the fractional B-T equation; in [7], Cenesiz et al. used the modified Taylor collocation method; and in [8], Krishnasamy and Razzaghi used the fractional Taylor method.The majority of FDEs do not yield an accurate solution.

* Corresponding author at King Salman International University

E-mail address: mostafa.taema@ksiu.edu.eg (M.A. Taema)

Thus, a variety of numerical techniques, including spectral techniques [9], differential transform techniques [10], and finite element techniques [11], have been used to generate approximate solutions to the FDEs. Spectral approaches are the most crucial techniques. There are three variations of them: the Tau, Collocation, and Galerkin methods. Numerous equations, such as partial differential equations, ordinary differential equations, and FDEs, can be solved using spectral methods. Numerous writers have made extensive use of these versions, such as Abd-Elhameed and Youssri [12], who presented an approximation method based on the tau method for solving coupled systems of FDEs. A collocation approach was used by Abd-Elhameed et al. [13] to resolve second-order nonlinear two-point boundary value issues. In order to tackle third-order linear two-point boundary value problems, Abd-Elhameed [14] used the Galerkin approach. Chebyshev polynomials are well-known for their optimal approximation properties and are frequently employed in spectral methods due to their fast convergence and numerical stability. While the first and second kinds of Chebyshev polynomials are widely studied, the fourth-kind Chebyshev polynomials offer

Previous Fibonacci-based method. The results confirm that distinct orthogonality properties and are particularly effective for dealing with certain types of boundary conditions. These features make them suitable for use in fractional differential equations like the Bagley-Torvik equation, where maintaining numerical precision is critical.

The primary objective of this study is to develop a robust numerical algorithm based on Chebyshev polynomials of the fourth kind and apply it to solve the fractional Bagley-Torvik equation efficiently. To achieve this, the non-homogeneous boundary conditions of the equation are first transformed into homogeneous ones, making it compatible with the polynomial approximation framework. The solution is then expanded in terms of Chebyshev polynomials, and the tau method is applied to convert the fractional differential equation into a system of algebraic equations. These equations are solved using Gaussian elimination, providing a computationally feasible solution even for large systems.

The paper also present several numerical examples to validate the effectiveness of the Chebyshev-based approach, comparing its performance against the Chebyshev polynomials not only enhance convergence but also offer greater numerical stability, particularly for higher-degree approximations .

This study contributes to the growing body of research on fractional calculus and numerical methods, offering an improved solution technique for the Bagley-Torvik equation. Furthermore, the proposed Chebyshev operational tau method has the potential to be extended to other types of fractional differential equations, expanding its application in various scientific and engineering fields .This work is structured as follows: introduces key mathematical properties of Chebyshev polynomials of the fourth kind, explains their importance in the tau method, and highlights the computational advantages they offer in section 1. In section 2, we go into function approximation and the Chebyshev spectral collocation technique. In section 3, a Chebyshev polynomials of the fourth kind for solving Bagley-Torvik equation is analyzed and presented .Section 4 covers the illustrated examples that demonstrate the correctness and efficiency of the current approach. Section 5 concludes by summarizing the research result.

2. Chebyshev polynomial of fourth-kind and its properties

The fourth-kind Chebyshev polynomials are introduced in this section, along with their features that are essential to their use in the operational tau approach for solving the fractional Bagley-Torvik equation. In contrast to the more widely used Chebyshev polynomials of the first and second kinds, the fourth-kind Chebyshev polynomials, represented by $W_n(x)$, have unique orthogonality and weighting functions that enable them to be used for approximating functions in certain situations, especially those where boundary behavior and numerical stability are crucial.

The Chebyshev polynomials of the fourth-kind $W_n(x)$, have unique orthogonality and weighting functions that , are defined using trigonometric expressions similar to other kinds of Chebyshev polynomials but with a distinct phase shift. They can be expressed as [15].

$$W_n(x) = \sqrt{\frac{2}{1-x}}\sin\left(\left(n+\frac{1}{2}\right)\ \arccos x\right), \qquad (2.1)$$

With a cosine function altered by a $\frac{n\pi}{2}$ phase shift, this formula emphasizes the polynomials' oscillating nature. The orthogonality features of the fourth-kind polynomials differ significantly from those of their first and second counterparts due to this phase shift.

In spectral techniques, the orthogonality of fourth-kind Chebyshev polynomials is essential because it guarantees that expanding the solution in terms of these polynomials minimizes the residual error in the best possible way. Over the interval $[-1,1]$, the fourth-kind Chebyshev polynomials are orthogonal with respect to a particular weight function, as shown by [15]:

$$\int_{-1}^{1} W_n(x)\, W_m(x)\, w(x)dx = 0, \quad \text{for} n \neq m\ , \qquad (2.2)$$

where $w(x)$ is the weight function associated with these polynomials, which is defined as[15]:

$$w(x) = \sqrt{\frac{1-x}{1+x}}. \qquad (2.3)$$

This weight function is essential to the precision of the polynomial approximations and is different from those applied to the first and second types of Chebyshev polynomials . As the number of terms in the series expansion rises, any approximation produced using the polynomials will converge optimally due to their orthogonality.

Recurrence relations are crucial for the computational application of polynomial-based techniques because they offer an alternative to computing higher-order polynomials directly from their trigonometric definitions. The fourth-kind Chebyshev polynomials satisfy the recurrence relation as follows [15]:

$$W_{n+1}(x) = 2x\, W_n(x) - W_{n-1}(x), \qquad (2.4)$$

for $n \geq 1$, with the initial conditions[15]:

$$W_0(x) = 1, \quad W_1(x) = 2x + 1.$$

For big n, this recurrence relation makes it possible to compute $W_n(x)$ efficiently, which is especially helpful for resolving differential equations that need for high-order polynomials in order to produce accurate approximations.When resolving differential equations, when precise approximations necessitate the use of high-order polynomials, this recurrence relation makes it possible to compute $W_n(x)$ efficiently for large n.

Chebyshev polynomials are used in the tau method because of their better approximation qualities. Particularly for issues involving fractional derivatives, like the Bagley-Torvik equation, these polynomials provide an ideal foundation for the spectrum representation of differential equation solutions.

In the operational tau method, the solution u(x) of the fractional differential equation is approximated by a series expansion in terms of Chebyshev polynomials of the fourth-kind:

$$u(x) \approx \sum_{i=0}^{M} c_i\, W_i(x), \tag{2.5}$$

Where the unknown coefficients to be found are denoted by c_i. By applying orthogonality criteria to the residual with respect to the Chebyshev basis, the tau method minimizes the differential equation's residual over the interval.

3. The fractional B-T equation's numerical approach and basis function selection

This section will develop the fractional Bagley-Torvik equation, which is a differential equation of second order that has both fractional and integer derivatives. When describing the motion of a rigid plate in a viscoelastic or Newtonian fluid, where the dynamics are controlled by both elastic and damping forces, the Bagley-Torvik equation is widely used. The memory-dependent, non-local behavior of viscoelastic materials is captured by the fractional order of the derivatives. The fractional Bagley-Torvik equation can be written as[16]:

$$a_1\, D^2\, v(x) + a_2\, D^{\frac{3}{2}}\, v(x) + a_3\, v(x) = h(x), \quad x \in (0, \ell), \tag{3.1}$$

where a_1, a_2 and a_3 are constants representing the physical properties of the system, $v(x)$ is the unknown function to be solved, $h(x)$ is a source term, and x represents the spatial variable over the domain $[1, \ell]$.

Boundary conditions confine the solution of the Bagley-Torvik equation in realistic physical models. Non-homogeneous boundary conditions at the domain's ends $x = 0$ and $x = \ell$ are typically included with the equation. The following is the format of these boundary conditions::

$$v(0) = v_0, \quad v(\ell) = v_\ell. \tag{3.2}$$

In which the boundary values v_0 and v_ℓ are provided. The direct application of spectral methods, such as the tau approach, is complicated by non-homogeneous boundary circumstances because the polynomials are often designed for homogeneous boundary conditions. The problem is transformed into one with homogeneous boundary conditions in order to get around this.

In order to use the tau approach with Chebyshev polynomials, we first convert the original equation with non-homogeneous boundary conditions into a homogeneous problem. To do this, a new function $u(x)$ that takes into consideration the boundary conditions is introduced, so that [16]:

$$v(x) = u(x) - \left(1 - \frac{x}{\ell}\right) v(0) - \frac{x}{\ell} v(\ell), \tag{3.3}$$

The function $u(x)$ is chosen so that it satisfies homogeneous boundary conditions:

$$u(0) = u(\ell) = 0, \tag{3.4}$$

Substituting this transformation into the original Bagley-Torvik equation, we obtain a modified form of the equation in terms of $u(x)$:

$$a_1\, D^2\, u(x) + a_2\, D^{\frac{3}{2}}\, u(x) + a_3\, u(x) = k(x), \tag{3.5}$$

where $k(x)$ is a modified source term that incorporates the effects of the original boundary conditions and the source term $h(x)$. Specifically, $k(x)$ is given by:

$$k(x) = h(x) - a_1 \left(1 - \frac{x}{\ell}\right) v_0 - a_2 \left(1 - \frac{x}{\ell}\right)^{\frac{3}{2}} v_0 - a_3 \left(1 - \frac{x}{\ell}\right)^2 v_0. \tag{3.6}$$

This formulation allows us to proceed with the tau method while preserving the boundary conditions inherent in the physical problem.

Definition 1.1 As shown in Podlubny [3], the Caputo definition of the fractional-order derivative is defined as:

$$D^\beta h(x) = \frac{1}{\Gamma(k-\beta)} \int_0^x (x-t)^{k-\beta-1}\, h^{(k)}(t)\, dt, \quad \beta > 0, x > 0, \quad k-1 \le \beta < k, \quad k \in \mathbb{N}, \tag{3.7}$$

The following properties are satisfied by the operator D^β for $k - 1 \le \beta < k$, $k \in \mathbb{N}$:

$$D^\beta C = 0 \qquad (C \;\; is \;\; a \;\; constant), \tag{3.8}$$

$$D^{\beta}x^{k}=\begin{cases}0, & \text{if} k\in\mathbb{N}_0 \text{and} k<[\beta]\\ \frac{\Gamma(k+1)}{\Gamma(k+1-\beta)}x^{k-\beta}, & \text{if} k\in\mathbb{N}_0 \text{and} k\geq[\beta]\end{cases} \tag{3.9}$$

Where $\mathbb{N}=\{1,2,\dots\}$ and $\mathbb{N}_0=\{0,1,2,\dots\}$. In addition, the notation $[\beta]$ represents the ceiling function.

As explained in Section 2, after the issue is converted into an equation with homogeneous boundary conditions, we use a series expansion in terms of fourth-kind Chebyshev polynomials to approximate the answer $u(x)$. The approximate answer is as follows:

Substituting the polynomial approximation of $u(x)$ into the transformed Bagley-Torvik equation, we obtain the residual $R(x)$

$$R(x)=a_1\sum_{i=0}^{M}c_i\,D^2\,W_i(x)+a_2\sum_{i=0}^{M}c_i\,D^{\frac{3}{2}}\,W_i(x)+a_3\sum_{i=0}^{M}c_i\,W_i(x)-k(x), \tag{3.10}$$

We can write the fourth-kind Chebyshev polynomials in form [17]

$$W_i(x)=\sum_{k=0}^{n}\frac{(-1)^{k+i}\,(2i+1)\,\Gamma(i+k+1)}{\sqrt{h_i}\,\Gamma\left(\frac{3}{2}\right)(i-k)!\,\Gamma\left(k+\frac{1}{2}\right)}x^k, \tag{3.11}$$

Where

$$h_i=\frac{(2i+1)\,\Gamma(i+1)\,\Gamma(i+3/2)}{i!\,\Gamma\left(\frac{3}{2}\right)^2\Gamma\left(i+\frac{1}{2}\right)}, \tag{3.12}$$

make some simplify we get

$$W_i(x)=\sqrt{\frac{1}{i!\left(i+\frac{1}{2}\right)}}\sum_{k=0}^{i}\frac{(-1)^{k+i}\,\Gamma(i+k+1)}{(i-k)!\,\Gamma\left(k+\frac{1}{2}\right)}x^k. \tag{3.13}$$

Theorem 1 [17] The following fractional derivative formulas hold if β is in the interval $]1,2]$.

$$D^{3/2}W_i(x)=\sqrt{\frac{1}{i!\,(i+\frac{1}{2})}}\sum_{k=0}^{i}\frac{(-1)^{k+i}\,\Gamma(i+k+1)\,\Gamma(k+1)}{(i-k)!\,\Gamma\left(k+\frac{1}{2}\right)\Gamma(k-\frac{3}{2}+1)}x^{k-\frac{3}{2}}, \tag{3.14}$$

where

$$D^{3/2}x^k=\frac{\Gamma(k+1)}{\Gamma(k-\frac{3}{2}+1)}x^{k-\frac{3}{2}}. \tag{3.15}$$

Now we will make second derivative for fourth-kind Chebyshev polynomials

$$D^2W_i(x)=\sqrt{\frac{1}{i!\left(i+\frac{1}{2}\right)}}\sum_{k=0}^{i}\frac{(-1)^{k+i}\,k(k-1)\,\Gamma(i+k+1)}{(i-k)!\,\Gamma\left(k+\frac{1}{2}\right)}x^{k-2}. \tag{3.16}$$

Now we will substitute (3.13), (3.14) and (3.16) in (3.10) we get

$$\begin{aligned}R(x)&=a_1\sum_{i=0}^{M}c_i\left(\sqrt{\frac{1}{i!\left(i+\frac{1}{2}\right)}}\sum_{k=0}^{i}\frac{(-1)^{k+i}\,k!\,\Gamma(i+k+1)}{(i-k)!\,\Gamma\left(k+\frac{1}{2}\right)}x^{k-2}\right)\\&+a_2\sum_{i=0}^{M}c_i\left(\sqrt{\frac{1}{i!\,(i+\frac{1}{2})}}\sum_{k=0}^{i}\frac{(-1)^{k+i}\,\Gamma(i+k+1)\,\Gamma(k+1)}{(i-k)!\,\Gamma\left(k+\frac{1}{2}\right)\Gamma(k-\frac{3}{2}+1)}x^{k-\frac{3}{2}}\right)\\&+a_3\sum_{i=0}^{M}c_i\left(\sqrt{\frac{1}{i!\left(i+\frac{1}{2}\right)}}\sum_{k=0}^{i}\frac{(-1)^{k+i}\,\Gamma(i+k+1)}{(i-k)!\,\Gamma\left(k+\frac{1}{2}\right)}x^k\right)-k(x).\end{aligned} \tag{3.17}$$

The application of the tau method yields

$$\int_0^{\ell}R(x)\,W_i(x)\,dx=0,\quad 0\leq i\leq M. \tag{3.18}$$

In the unknown coefficients ci, Eq. 3.17 creates a system of algebraic equations that can be resolved using the Gaussian elimination method.

4 Illustrative Examples

The current process is demonstrated using a few numerical examples to demonstrate the practicality and reliability of the aforementioned technique.

Example 1 *.As given in Mdallal et al.[18],consider the fractional B-T equation*

$$D^{(2)}v(x)+D^{(\frac{3}{2})}v(x)+v(x)=2+4\sqrt{\frac{x}{\pi}}+x^2,\quad x\in(0,5). \tag{4.1}$$

subject to

$$v(0)=0,\quad v(5)=25. \tag{4.2}$$

where the exact solution is $v(x)=x^2$. Using the transformation (3.3) in Equ.(4.3) and Equ. (4.2) we will get

$$D^{(2)}u(x)+D^{\left(\frac{3}{2}\right)}u(x)+u(x)=2-5x+4\sqrt{\frac{x}{\pi}}+x^2,\quad x\in(0,5), \tag{4.3}$$

subject to

$$u(0)=0,\quad u(5)=0. \tag{4.4}$$

and the exact solution $u(x)=x^2-5x$. We will convert the interval from (0,5) to (-1,1) then apply the present method to Eq. 4.3 .

Table 1: Our Method of Example 1 at $M=2$ and $M=3$

M	2	3
Our Method	0	4.14165×10^{-17}

Example 2 [16] Consider the fractional Bagley-Torvik Equation

$$D^{(2)}v(x)+D^{\left(\frac{3}{2}\right)}v(x)+v(x)=e^{\gamma x}\left[1+\gamma^2+\gamma^{\frac{3}{2}}\operatorname{erf}(\sqrt{\gamma x})\right],\quad x\in(0,1), \tag{4.5}$$

subject to

$$v(0)=1,\quad v(1)=e^{\gamma}, \tag{4.6}$$

whose exact solution is given by $v(x)=e^{\gamma x}$ where $\operatorname{erf}(x)$ is defined as:

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}\,dy. \tag{4.7}$$

Table 2: Campare result of Example 2 at $\gamma=1$

M	2	3	4	5	6
Our methos	1.77544×10^{-2}	4.13027×10^{-3}	1.40042×10^{-4}	4.7002×10^{-6}	1.40197×10^{-7}

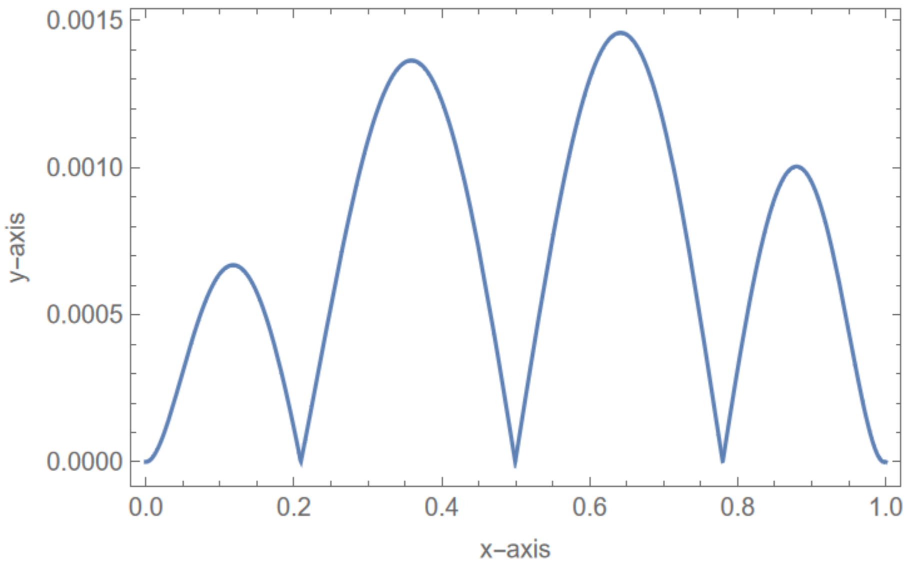

Figure 1: The absolute error at M=6 and $\gamma = 1$ of Example 2 .

Table 3: MAE of Example 2 at $\gamma = \pi$

M	2	4	6	8
Our methos	1.50159	0.140851	0.00130219	1.16051×10^{-5}

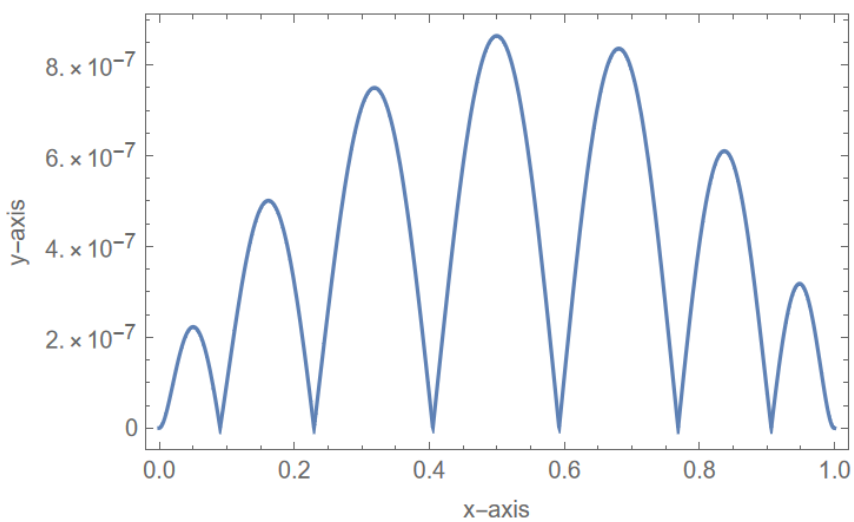

Figure 2: The absolute error at M=9 and $\gamma = 1$ of Example 2 .

Example 3 [8, 19] Consider the the fractional Bagley-Torvik equation

$$D^{(2)}v(x) + \frac{8}{17}D^{\left(\frac{3}{2}\right)}v(x) + \frac{13}{51}v(x) = \frac{x^{-\frac{1}{2}}}{89250\sqrt{\pi}}\left(48g(x) + 7\sqrt{\pi x}f(x)\right), \quad x \in]0,1], \tag{4.8}$$

subject to

$$v(0) = 0, \quad v(1) = 0, \tag{4.9}$$

where

$$g(x) = 16000x^4 - 32480x^3 + 21280x^2 - 4746x, \tag{4.10}$$

and

$$f(x) = 3250x^5 - 9425x^4 + 264880x^3 - 448107x^2 + 233262x - 34578. \tag{4.11}$$

The exact solution of Eq. 4.8 is

$$v_1(x) = x^5 - \frac{29}{10}x^4 + \frac{76}{25}x^3 - \frac{339}{250}x^2 + \frac{27}{125}x. \tag{4.12}$$

Table 4: Our method of Example 3 at M=5 and M=7

M	5	7
Our methos	0	1.06297×10^{-15}

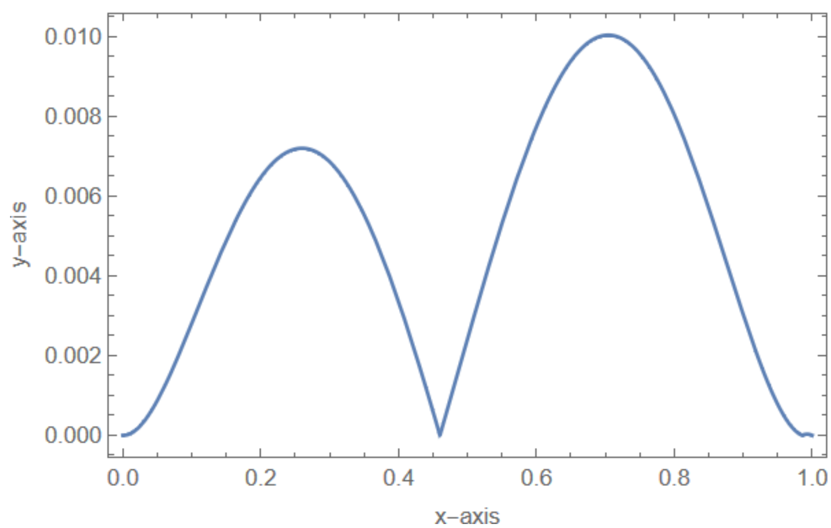

Figure 3: The absolute error at M=4 of Example 3 .

Conclusion

In this paper, we introduced a reliable numerical method for solving the fractional Bagley-Torvik equation with fourth-kind Chebyshev polynomials. We converted the original problem into a system of algebraic equations that could be solved by utilizing the characteristics of these orthogonal polynomials and their fractional derivatives. The solution process was made much easier by the operational tau technique, which guaranteed precision and computing effectiveness.

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