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Optimal Control Problem of Hybrid FracInt Variable-Order Mathematical Model for Covid-19 with Time Delay

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ABSTRACT

This paper investigates the variable-order FracInt derivative with time delay to examine the corona-virus disease. In the present research, an issue with optimal control is presented for the combined variable-order Coronavirus FracInt Model with time delay. The suggested FracInt approach is successful for controlling the disease, according to the numerical results. The Caputo derivative and the Riemann-Liouville integral are combined linearly to yield the derivative of hybrid fractional order. The definition of the variable-order derivatives is fracInt order derivatives. We provide the solutions' existence, boundedness, positivity, and reproductive number R_0. Two control factors, v_1 and v_2 are thought to lessen the risk of infection spreading to healthy people. The new hybrid variable-order operator can be approximated using the Grünwald-Letnikov approximation. The optimality system is solved using a hybrid variable-order operator combined with the non-standard finite difference approach. Lastly, the theoretical analysis of the proposed model has been supported by numerical simulations.

1. Introduction

The corona-virus disease (COVID-19) is still a major global health concern that has an adverse effect on Egypt's economy in addition to destroying a great deal of people and causing large financial losses. On February 14, 2020, the Egyptian authorities reported the first instance of the new corona-virus based on publicly accessible data. Epidemic disease prevention largely depends on our ability to comprehend the mechanics of disease transmission and develop efficient control measures. Numerous governments were compelled to impose strategies that appeared outdated, such isolating individuals and overthrowing entire areas or even countries. These actions threaten individual rights, harm the business and economic sectors, and present a serious threat to employment. Face masks have been found to be crucial for managing the corona-virus disease. It plays a significant part in corona-virus disease control by lowering the chance of the illness spreading further ([12], [27]). To protect people from the corona-virus, medical professionals and the World Health Organization (WHO) offered advice on how to avoid social situations where the virus could spread. These included wearing face masks, cleaning or washing hands, and avoiding close contact with others.

Recently, several Numerical models have been constructed to explain the behavior of a corona-virus infection. The majority of these models rely on classical derivatives of integer orders, which do not account for the fading memory characteristic of many biological processes. On the other hand, variable-order fractional derivatives can explain the genetic and memory characteristics of different materials and processes. Consequently, the fractional derivatives of variable order are useful and appropriate for use in epidemic models. The applications of epidemic models based on variable-order fractional derivatives have shown some advantages over models built with integer-order derivatives and have greater ability to produce precise conclusions from actual data ([13], [21], [26]). The literature has investigated and examined a wide range of mathematical models to study the intricate patterns of transmission of the current COVID-19 pandemic ([5], [10], [11], [34]).

Accurately solving differential equations with fractional variable orders analytically is an extremely challenging task. For the objective of calculating the solutions of these models, computational method research is important. The fractional differential equations of variable order and their

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solution has been approximated by the finite difference techniques (FDM), which have undergone significant updates. Fractional derivative-based mathematical models provide more information about the diseases under investigation ([4], [7], [8], [20], [23], [30]). The fractional hybrid operator can be obtained by combining linearly the Caputo fractional derivative and the fractional integral of Riemann-Liouville. Compared to The fractional operator of Caputo, it is one of the most adaptable, reliable, and efficient operators. Therefore, when describing biological phenomena, this operator is more appropriate than the Caputo operator ([9], [15], [29]).

Many sectors, including biology, have seen an increase in the use of fractional optimal control, or (FOC), for the treatment of disease. (FOC) involves minimizing (or maximizing) an objective functional subject to dynamic constraints on control and state variables, where these conditions include derivatives of fractional order. There was discussion of several numerical techniques for estimating the solutions for particular kinds of fractional optimal control problems ([1], [2], [6], [14]). Sweilam and Al-Mekhlafi recently presented a number of numerical research on models of variable-order optimal control, for more details see ([24], [25], [31]).

In this work, we focused particularly on the investigation of the Covid-19 variable-order FracInt mathematical model and its optimal control with time delay. We are going to investigate the existence, boundedness, positivity, and basic reproductive number R_0 of the current model. Both the Caputo variable-order derivative and the Riemann-Liouville variable-order integral are used to define the variable-order derivatives. It is thought that control variables v_1 and v_2 will reduce the spread of infection to healthy individuals. The optimality system can be resolved numerically by utilizing the Non-Standard Finite Difference approach and the Grünwald-Letnikov approximation (CPC-GLNFDM) with a hybrid variable-order operator. Our numerical examples demonstrate the value of our results by providing strong evidence for the theoretical results.

This article's remaining sections are arranged as follows: Section (2) introduces some Variable order fractional derivative definitions and notations. The model that includes control variables and a hybrid variable-order FracInt with time delay is described in Section (3). Additionally, the existence, boundedness, positivity, and basic reproductive number R_0 of the solutions are discussed. Section (4) provides the hybrid variable-order optimal control formulation for the suggested model. The numerical approach for non-standard finite differences and the Grünwald-Letnikov approximation is developed, and Section (5) studies The proposed method's stability analysis. The proposed model's numerical simulations are displayed in Section (6) to demonstrate the usefulness and efficiency of the proposed approach. Finally, a summary of the conclusions may be found in Section (7).

2. Basic Definitions

Definition 2.1 *We assume that:* $\Omega = [a, b]$, $-\infty < a < b < +\infty$, $\alpha \in \mathbb{C}$, *the sides on the left and right Caputo's derivatives for a function f(t) of order* α , $f \in AC^n[a, b]$ *are determined by [16]:*

$$({}^C D^\alpha_{a+} f)(t) = ({}^C_a D^\alpha_t f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^n(s)}{(t-\xi)^{1-n+\alpha}} ds, \quad t > a, \tag{1}$$

$$({}^C D^\alpha_{b-} f)(x) = ({}^C_t D^\alpha_b f)(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b \frac{f^n(s)}{(s-t)^{1-n+\alpha}} ds, \quad t < b. \tag{2}$$

Where $n = [\Re(\alpha)] + 1$, $\Re(\alpha) \notin \mathbb{N}_0$.

Definition 2.2 *We assume that:* $\Omega = [a, b]$, $-\infty < a < b < +\infty$, $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$, *the sides on the left and right For a continuous function* $f(t)$, *the Riemann-Liouville's derivatives of order* α *are determined by [16]:*

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(s)}{(t-s)^{1-n+\alpha}} ds, \quad t > a, \tag{3}$$

$${}_tD_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{-d}{dt}\right)^n \int_t^b \frac{f(s)}{(s-t)^{1-n+\alpha}} ds, \quad t < b. \tag{4}$$

Where $n = [\Re(\alpha)] + 1$

Definition 2.3 *Let* $\Omega = [a, b]$, $-\infty < a < b < +\infty$, $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$, *for a continuous function* $f(t)$, *the left-side and right-side Riemann-Liouville's integrals of order* α *are defined, respectively, by [16]:*

$${}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \left[\int_a^t f(s)(t-s)^{\alpha-1} ds\right], \qquad t > a, \tag{5}$$

$${}_tI_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \left[\int_t^b f(s)(t-s)^{\alpha-1} ds\right], \qquad t < b. \tag{6}$$

Where $0 < \alpha < 1$

Proposition 2.1 *[16] Let* ${}_aD_t^\alpha f$, ${}_tD_b^\alpha f$ *are the fractional derivatives of the Riemann-Liouville's left and right-side* $f(t)$ *and*

${}_{a}^{C}D_{t}^{\alpha}f$, ${}_{t}^{C}D_{b}^{\alpha}f$ *are the fractional derivatives of Caputo's left and right-side* $f(t)$, $\alpha \notin \mathbb{N}$, *then for* $n = [\Re(\alpha)] + 1$,

$${}_{a}D_{t}^{\alpha}f(t) = {}_{a}^{C}D_{t}^{\alpha}f(t) + \sum_{k=0}^{n-1} \frac{(t-a)^{(k-\alpha)}}{\Gamma(k-\alpha+1)} f^{(k)}(a), \quad (7)$$

$${}_{t}D_{b}^{\alpha}f(t) = {}_{t}^{C}D_{b}^{\alpha}f(t) + \sum_{k=0}^{n-1} \frac{(b-t)^{(k-\alpha)}}{\Gamma(k-\alpha+1)} f^{(k)}(b). \quad (8)$$

Definition 2.4 *[9] The general definition of Caputo proportional fractional hybrid operator (CP) is given by:*

$$\begin{aligned} {}_{0}^{CP}D_{t}^{\alpha}y(t) &= \left(\int_0^t \left(y(s)K_1(s,\alpha) + y'(s)K_0(s,\alpha)\right)(t-s)^{-\alpha} ds\right) \frac{1}{\Gamma(1-\alpha)} \\ &= \left(K_1(t,\alpha)y(t) + K_0(t,\alpha)y'(t)\right)\left(\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\right). \end{aligned} \quad (9)$$

Where, $K_0(\alpha,t) = \alpha t^{(1-\alpha)}$, $K_1(\alpha,t) = (1-\alpha)t^{\alpha}$, $0 < \alpha < 1$.

Definition 2.5 *The definition of the Caputo proportional constant fractional hybrid operator, or CPC, is as follows [9]:*

$$\begin{aligned} {}_{0}^{CPC}D_{t}^{\alpha}y(t) &= \left(\int_0^t (t-s)^{-\alpha}\left(y(s)K_1(\alpha) + y'(s)K_0(\alpha)\right) ds\right) \frac{1}{\Gamma(1-\alpha)} \\ &= K_1(\alpha){}_{0}^{RL}I_{t}^{1-\alpha}y(t) + K_0(\alpha){}_{0}^{C}D_{t}^{\alpha}y(t), \end{aligned} \quad (10)$$

where, $K_0(\alpha) = \alpha Q^{(1-\alpha)}$, $K_1(\alpha) = (1-\alpha)Q^{\alpha}$, Q is a constant.

Rosa and Torres, recently introduced a set of deferential equations; also, it is classical or fractional, with the derivatives' order changing over time [18]. There are various advantages to this theory. It has been discovered that one such system, a fractional system with variable order called the FractInt system which is referred to Herien, is helpful to controlling the illness. The fractional-integer (FractInt) system's derivative order changes correspondingly:

$$\alpha(t) = \begin{cases} \alpha_0 = 1 & t' \geq t \geq 0, \\ 0 < \alpha_1 \leq 1, & T_f \geq t > t'. \end{cases}$$

3. FracInt Variable-Order of Mathematical Model with Time Delay

We developed The COVID-19 model which presented in [27] to a time-delayed hybrid FracInt variable-order fractional model. In order to decrease the spread of infection among healthy individuals, two control variables, v_1 and v_2, are include; v_1 reflects preventive actions such social separation and regular usage of face masks. The isolation rate for infected people who are not hospitalized at a given period is represented by v_2. To ensure that the resulting variable-order fractional equations' two sides are dimensionally matched, an auxiliary parameter μ is added to the variable-order fractional operator. This means that the left side has a dimension of day^{-1} [32].

$$\begin{aligned}
&\mu^{\alpha(t)-1}{}_{0}^{CPC}D_{t}^{\alpha(t)}S = \beta(1-v_1)I\frac{S}{N} - \beta(1-v_1)I(t-\tau)\frac{S(t-\tau)}{N}e^{-\mu\tau} - \beta\eta A\frac{S}{N}, \\
&\mu^{\alpha(t)-1}{}_{0}^{CPC}D_{t}^{\alpha(t)}E = \beta(1-v_1)I(t-\tau)\frac{S(t-\tau)}{N}e^{-\mu\tau} + \beta\eta A\frac{S}{N} - \sigma E, \\
&\mu^{\alpha(t)-1}{}_{0}^{CPC}D_{t}^{\alpha(t)}I = \alpha_1\sigma E - \gamma_I I - \phi I - r_1 v_2 I, \\
&\mu^{\alpha(t)-1}{}_{0}^{CPC}D_{t}^{\alpha(t)}A = (1-\alpha_1)\sigma E - \gamma_A A, \\
&\mu^{\alpha(t)-1}{}_{0}^{CPC}D_{t}^{\alpha(t)}H = \phi I - \gamma_H H - \delta H, \\
&\mu^{\alpha(t)-1}{}_{0}^{CPC}D_{t}^{\alpha(t)}R = \gamma_A A + \gamma_I I + \gamma_H H + r_1 v_2 I, \\
&\mu^{\alpha(t)-1}{}_{0}^{CPC}D_{t}^{\alpha(t)}D = \delta H,
\end{aligned} \quad (12)$$

$$R + I + A + E + S = N,$$

and the initial conditions are:

$$I(0) = i_0 \geq 0, \quad E(0) = e_0 \geq 0, \quad A = a_0 \geq 0, \quad S(0) = s_0 \geq 0, \quad H = h_0 \geq 0, \quad R(0) = r_0 \geq 0, \quad D = d_0 \geq 0. \quad (13)$$

Where (D) indicates the cumulative deaths, (R) is the recovered class, (I) is the symptomatic infectious class, (S) is the susceptible class, (H) is the hospitalized class, (E) is the exposed class, and The hospitalized asymptomatic infectious class is denoted by (A). It is assumed that a certain percentage of infected individuals who exhibit symptoms and enter the hospitalized H group is not able to spread the illness to other people [12].

Table (1): [27] The specification of each system parameter (12).

Name	the definition	Value (per day^{-1})	References
σ	Transition exposed to infectious	$(1/1.5)$	[12]
η	Transmission coefficient because of super-spreaders	1.9 dimensionless	[27]
β	Infectious contact rate	2.5	[27]
γ_A	Recovery rate, Asymptomatic	$(1/7)$	[12]
γ_H	Recovery rate, hospitalized	$(1/14)$	[12]
δ	The percentage of an infected class's death due to disease	0.015	[12]
γ_I	Recovery rate, symptomatic	$(1/7)$	[12]
α_1	A portion of illnesses that show symptoms	0.5 dimensionless	[12]
γ	The recovery rate	$(1/7)$	[12]
ϕ	rate of hospitalization	0.0025	[12]

3.1 Boundedness and Positivity

By involving all of the system's equations, The model solution's boundedness is confirmed in the manner described below (12):

$$ {}_0^{CPC}D_t^{\alpha(t)} N_H(t) = 0, \quad N_H(0) = A \geq 0, \tag{14}$$

where the complete summation of the population in (12) is denoted by N_H and A is constant, the equation (14) has the following solution [9]:

$$ N_H(t) \geq A e^{(-\frac{K_1(\alpha(t))}{K_0(\alpha(t))}t)}, \tag{15}$$

the inequality (15) implies that $\lim_{t\to\infty} N_H(t) \geq A$. Consequently, the conclusion $N_H(t) \geq 0$, at $t \longrightarrow \infty$. So, the system (12) has bounded solutions.

Lemma 3.1 *For the initial conditions (13), all of the system's solutions (12) stay nonnegative for* $t \geq 0$.

Proof. It was found that according to the initial conditions (13):

$$\begin{aligned}
&\mu^{\alpha(t)-1} {}_0^{CPC}D_t^{\alpha(t)} S|_{S=0} = 0 \geq 0,\\
&\mu^{\alpha(t)-1} {}_0^{CPC}D_t^{\alpha(t)} E|_{E=0} = \beta(1-v_1)I(t-\tau)\frac{S(t-\tau)}{N}e^{-\mu\tau} + \beta\eta A\frac{S}{N} \geq 0,\\
&\mu^{\alpha(t)-1} {}_0^{CPC}D_t^{\alpha(t)} I|_{I=0} = \alpha_1\sigma E \geq 0,\\
&\mu^{\alpha(t)-1} {}_0^{CPC}D_t^{\alpha(t)} A|_{A=0} = (1-\alpha_1)\sigma E \geq 0,\\
&\mu^{\alpha(t)-1} {}_0^{CPC}D_t^{\alpha(t)} H|_{H=0} = \phi I \geq 0,\\
&\mu^{\alpha(t)-1} {}_0^{CPC}D_t^{\alpha(t)} R|_{R=0} = \gamma_A A + \gamma_I I + \gamma_H H + r_1 v_2 I \geq 0,\\
&\mu^{\alpha(t)-1} {}_0^{CPC}D_t^{\alpha(t)} D|_{D=0} = \delta H \geq 0.
\end{aligned} \tag{16}$$

3.2 Existence and Basic Reproductive Number of the Suggested Model

In this instance, the fixed point theory will be used. Let us write the following for the system (12):

$$ {}_0^{CPC}D_t^{\alpha(t)} y(t) = Q(y(t),t), \quad y(0) = y_0 \geq 0. \tag{17}$$

$y(t) = (H, I, S, D, A, E, R, v_1(t), v_2(t))^T$represents the state variables, where Q is a continuous function's vector that contains the following:

$$\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \end{pmatrix} = \begin{pmatrix} (\beta(1-v_1)I\frac{S}{N} - \beta(1-v_1)I(t-\tau)\frac{S(t-\tau)}{N}e^{-\mu\tau} - \beta\eta A\frac{S}{N})\mu^{1-\alpha(t)} \\ (\beta(1-v_1)I(t-\tau)\frac{S(t-\tau)}{N}e^{-\mu\tau} + \beta\eta A\frac{S}{N} - \sigma E)\mu^{1-\alpha(t)} \\ (\alpha_1\sigma E - \gamma_I I - \phi I - r_1 v_2 I)\mu^{1-\alpha(t)} \\ ((1-\alpha_1)\sigma E - \gamma_A A)\mu^{1-\alpha(t)} \\ (\phi I - \gamma_H H - \delta H)\mu^{1-\alpha(t)} \\ (\gamma_A A + \gamma_I I + \gamma_H H + r_1 v_2 I)\mu^{1-\alpha(t)} \\ (\delta H)\mu^{1-\alpha(t)} \end{pmatrix},$$

under the initial condition y_0. Additionally, the quadratic vector function Q satisfies the Lipschitz condition, indicating the existence of: $G^0 \in \mathbb{R}$, such that [10]:

$$\| Q(y_1(t),t) - Q(y_2(t),t) \| \leq G^0 \| y_1(t) - y_2(t) \|. \quad (18)$$

The fundamental reproductive number R_0, which is determined by using the subsequent generation's matrix approach, is a crucial parameter for examining the model's threshold dynamics. It provides for a comprehensive Model of compartmentalized disease transmission based on the next generation matrix's spectral radius and the set of ordinary differential equations. The model is based on a series of standard equations that show how the number of people in each compartment changes over time. If $R_0 \leq 1$, the disease free equilibrium (DFE) is locally asymptotically stable; if $R_0 \geq 1$, on the other hand, there's always a chance of invasion because the DFE is unstable. R_0 is determined the following: The local stability of (DFE) is examined utilizing the operator technique of next generation [33]. As a result, we provide the matrices Accordingly, the new infection terms are represented by A, and the existing transfer terms connected to the baseline model are represented by B.

$$A = \mu^{1-\alpha(t)} \begin{pmatrix} 0 & \beta e^{-\mu\tau} & \beta\eta e^{-\mu\tau} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$B = \mu^{1-\alpha(t)} \begin{pmatrix} \sigma & 0 & 0 \\ -\alpha_1\sigma & (\gamma_I + \phi) & 0 \\ -(1-\alpha_1)\sigma & 0 & \gamma_A \end{pmatrix}.$$

The basic reproduction number of the model, denoted by R_0, is:

$$\rho(AB^{-1}) = R_0 = \mu^{1-\alpha(t)}\left(\frac{\beta\alpha_1}{(\phi+\gamma_I)} + \frac{\beta\eta(1-\alpha_1)}{\gamma_A}\right)e^{-\mu\tau}. \quad (19)$$

4 Delay of Optimal Control Problem

The system (12) is in $\mathbb{R}^7$, and

$$\Omega = \{(v_1(.), v_2(.)) | v_1, v_2(.) \text{ are Lebsegue measurable on } [0,1],$$

$$0 \leq v_1(.), v_2(.) \leq 1, \forall t \in [0, T_f]\},$$

be the set of acceptable controls. The objective functional will be defined as follows:

$$J(v_1, v_2) = \int_0^{T_f} (I(t) + B_1 v_1^2(t) + B_2 v_2^2(t))dt, \quad (20)$$

where, B_1 and B_2 are weight constants. Our goal is to find $v_1(t)$, $v_2(t)$ and the next cost functional is minimum:

$$J(v_1, v_2) = \int_0^{T_f} \eta(t, I, S, E, R, A, D, v_1, v_2)dt, \quad (21)$$

according to the constraints:

$${}_{a}^{CPC}D_t^{\alpha(t)}\Psi_j = \xi_i. \quad (22)$$

Where

$$\xi_i = \xi_i(t, S, I, E, R, A, D, v_1, v_2), \quad i, j = 1, \dots, 7,$$
$$\Psi_j = \{t, S, E, I, A, R, D, v_1, v_2\},$$

$\Psi_1(0) = S_0, \Psi_2(0) = E_0, \Psi_3(0) = I_0, \Psi_4(0) = A_0, \Psi_5(0) = R_0, \Psi_6(0) = D_0.$

We apply a modified form of Pontryagin's maximum principle in the variable order fractional issue. Agrwal presents this principles for fractional order [3], while some references ([24], [25]) use the caputo defintion for variable order. Here, we will numerically extend the necessary optimal control to hybrid variable order fractional derivatives in the way described below:

The definition of the Hamiltonian is:

$$H(S, A, E, I, R, D, t, v_1, v_2, \lambda_i) = \eta(t, A, S, I, E, D, R, v_1, v_2, \lambda_i) + \sum_{i=1}^{7} \lambda_i \xi_i(t, I, S, A, R, E, D, v_1, v_2). \quad (23)$$

Next, the following are the necessary optimal control conditions:

$$ {}^{CPC}_{t}D^{\alpha(t)}_{t_f} \lambda_\iota = \frac{\partial H}{\partial \vartheta_\iota}, \quad \iota = 1, \ldots, 7, \quad (24)$$

and,

$$\vartheta_\iota = \{t, S, I, E, A, R, D, v_1, v_2, \quad \iota = 1, \ldots, 7\},$$

$$0 = \frac{\partial H}{\partial v_k}, \quad k = 1,2, \quad (25)$$

$$ {}^{CPC}_{0}D^{\alpha(t)}_{t} \vartheta_\iota = \frac{\partial H}{\partial \lambda_\kappa}, \quad \iota = 1, \ldots, 7, \quad (26)$$

also, the lagrange multipliers must satisfy the following requirements:

$$\lambda_\iota(T_f) = 0, \quad \iota = 1,2, \ldots, 7. \quad (27)$$

$$H = \lambda_1[\beta(1 - v_1)I\frac{S}{N} - \beta(1 - v_1)I(t - \tau)\frac{S(t - \tau)}{N}e^{-\mu\tau} - \beta\eta A\frac{S}{N}] + \lambda_2[\beta(1 - v_1)I(t - \tau)\frac{S(t - \tau)}{N}e^{-\mu\tau}$$
$$+\beta\eta A\frac{S}{N} - \sigma E] + \lambda_3[\alpha_1\sigma E - \gamma_I I - \phi I - r_1 v_2 I] + \lambda_4[(1 - \alpha_1)\sigma E - \gamma_A A]$$
$$+\lambda_5[\phi I - \gamma_H H - \delta H] + \lambda_6[\gamma_A A + \gamma_I I + \gamma_H H + r_1 v_2 I] + \lambda_7[\delta H] + I + B_1 v_1^2 + B_2 v_2^2. \quad (28)$$

Adjoint variable order fractional differential equations [28]:

$$ {}^{CPC}_{t}D^{\alpha(t)}_{t_f} \lambda_1 = \frac{\partial H}{\partial S} + X[0, t_{f-\tau}]\frac{\partial H(t+\tau)}{\partial S(t-\tau)},$$

$$ {}^{CPC}_{t}D^{\alpha(t)}_{t_f} \lambda_3 = \frac{\partial H}{\partial I} + X[0, t_{f-\tau}]\frac{\partial H(t+\tau)}{\partial I(t-\tau)},$$

$$where: X[0, t_{f-\tau}] = \begin{cases} 1, & t \in [0, t_{f-\tau}] \\ 0, & otherwise. \end{cases} \quad (29)$$

Theorem 4.1 *There are an optimal control variables* v_1^*, v_2^* *with the corresponding solutions* S^*, E^*, I^*, A^*, R^*, D^*, *that minimizes* $J(v_1, v_2)$ *over* Ω. *Also, there exists adjoint variables* $\lambda_i, i = 1,2,3, \ldots, 7$, *satisfy the following:*

(i) adjoint equations

$$\mu^{\alpha(t)-1} {}^{CPC}_{t}D^{\alpha(t)}_{t_f} \lambda_1 = (\lambda_1\beta(1 - v_1^*)\frac{I^*}{N} - \lambda_1\beta\eta\frac{A^*}{N} + \lambda_2\beta\eta\frac{A^*}{N} - \lambda_1\beta(1 - v_1^*(t + \tau)\frac{I^*}{N}e^{-\mu\tau}$$
$$+\lambda_2\beta(1 - v_1^*(t + \tau)\frac{I^*}{N}e^{-\mu\tau}),$$

$$\mu^{\alpha(t)-1} {}^{CPC}_{t}D^{\alpha(t)}_{t_f} \lambda_2 = (-\sigma\lambda_2 + \alpha_1\sigma\lambda_3 + (1 - \alpha_1)\sigma\lambda_4),$$

$$\mu^{\alpha(t)-1} {}^{CPC}_{t}D^{\alpha(t)}_{t_f} \lambda_3 = (1 + \lambda_1\beta(1 - v_1^*)\frac{S^*}{N} - \phi\lambda_3 - \gamma_I\lambda_3 - r_1 v_2^*\lambda_3 + \phi\lambda_5 + \gamma_I\lambda_6 - r_1 v_2^*\lambda_6 +$$
$$+\lambda_1(-\beta(1 - v_1^*(t + \tau))\frac{S^*}{N}e^{-\mu\tau} + \lambda_2(\beta(1 - v_1^*(t + \tau))\frac{S^*}{N}e^{-\mu\tau}),$$

$$\mu^{\alpha(t)-1}{}_{t}^{CPC}D_{t_f}^{\alpha(t)}\lambda_4 = (-\frac{\beta\eta\lambda_1 S^*}{N} + \frac{\beta\eta\lambda_2 S^*}{N} - \gamma_A\lambda_4 + \gamma_A\lambda_6),$$

$$\mu^{\alpha(t)-1}{}_{t}^{CPC}D_{t_f}^{\alpha(t)}\lambda_5 = (-\delta\lambda_5 - \gamma_A\lambda_5 + \gamma_H\lambda_6 + \delta\lambda_7),$$

$$\mu^{\alpha(t)-1}{}_{t}^{CPC}D_{t_f}^{\alpha(t)}\lambda_6 = 0,$$

$$\mu^{\alpha(t)-1}{}_{t}^{CPC}D_{t_f}^{\alpha(t)}\lambda_7 = 0. \quad (30)$$

(ii) The transversality conditions are :

$$\lambda_\iota(T_f) = 0, \qquad \iota = 1,2,\ldots,7. \quad (31)$$

(iii) Conditions for optimality are :

$$H(E,S,I,A,R,D,v_1,v_2,\lambda,t) = \min_{0\le v_1,v_2\le 1} H(S,E,I,A,R,D,v_1,v_2,\lambda,t). \quad (32)$$

Moreover:

$$v_1^* = \frac{\partial H}{\partial v_1} = \min\{1, \max\{0, \frac{\lambda_1\beta I^* S^* + \beta I^*(t-\tau)S^*(t-\tau)e^{-\mu\tau}(\lambda_2-\lambda_1)}{2NB_1}\}\}, \quad (33)$$

$$v_2^* = \frac{\partial H}{\partial v_2} = \min\{1, \max\{0, \frac{r_1 I^*(\lambda_3-\lambda_6)}{2B_2}\}\}. \quad (34)$$

Proof. Equations (30) can be gained from (24), where:

$$H = \lambda_1^{CPC}{}_0D_t^{\alpha(t)}S^* + \lambda_2^{CPC}{}_0D_t^{\alpha(t)}E^* + \lambda_3^{CPC}{}_0D_t^{\alpha(t)}I^*$$
$$+\lambda_4^{CPC}{}_0D_t^{\alpha(t)}A^* + \lambda_5^{CPC}{}_0D_t^{\alpha(t)}H^* + \lambda_6^{CPC}{}_0D_t^{\alpha(t)}R^* + \lambda_7^{CPC}{}_0D_t^{\alpha(t)}D^*$$
$$I^* + B_1 v_1^{2*} + B_2 v_2^{2*}, \quad (35)$$

is the Hamiltonian. $\lambda_\kappa(T_f) = 0,\ \kappa = 1,\ldots,7,$ are hold. Equations $(33)-(34)$ is obtained from (32).

Currently, by substituting v_1^*, v_2^*, in (12), these are the state equations that we have [19]:

$$\mu^{\alpha(t)-1}{}_0^{CPC}D_t^{\alpha(t)}S^* = \beta(1-v_1^*)I^*\frac{S^*}{N} - \beta(1-v_1^*)I^*(t-\tau)\frac{S^*(t-\tau)}{N}e^{-\mu\tau} - \beta\eta A^*\frac{S^*}{N},$$
$$\mu^{\alpha(t)-1}{}_0^{CPC}D_t^{\alpha(t)}E^* = \beta(1-v_1^*)I^*(t-\tau)\frac{S^*(t-\tau)}{N}e^{-\mu\tau} + \beta\eta A^*\frac{S^*}{N} - \sigma E^*,$$
$$\mu^{\alpha(t)-1}{}_0^{CPC}D_t^{\alpha(t)}I^* = \alpha_1\sigma E^* - \gamma_I I^* - \phi I^* - r_1 v_2^* I^*,$$
$$\mu^{\alpha(t)-1}{}_0^{CPC}D_t^{\alpha(t)}A^* = (1-\alpha_1)\sigma E^* - \gamma_A A^*,$$
$$\mu^{\alpha(t)-1}{}_0^{CPC}D_t^{\alpha(t)}H^* = \phi I^* - \gamma_H H^* - \delta H^*,$$
$$\mu^{\alpha(t)-1}{}_0^{CPC}D_t^{\alpha(t)}R^* = \gamma_A A^* + \gamma_I I^* + \gamma_H H^* + r_1 v_2^* I^*,$$
$$\mu^{\alpha(t)-1}{}_0^{CPC}D_t^{\alpha(t)}D^* = \delta H^*. \quad (36)$$

5 Numerical Methods for the Suggested Model

5.1 CPC-GLNFDM

The hybrid fractional order derivatives equation is :

$$({}_0^{CPC}D_t^{\alpha(t)}y)(t) = \rho_0 y(t) + \rho_1 y(t-\Delta t), \quad t \ge 0, \quad 0 < \alpha(t) \le 1, \quad (37)$$
$$y(t) = \Psi(t), \quad t \in [-\tau, 0], \quad y(0) = y_0,$$

where, $\rho_0 < 0,\ \rho_1 < \rho_0$ and $\Psi(t)$ is bounded and continous function.

The formula for the equation (10) is as follows [22]:

$${}_0^{CPC}D_t^{\alpha(t)}y(t) = \frac{1}{\Gamma(1-\alpha(t))}\int_0^t (t-s)^{-\alpha(t)}(K_1(\alpha(t))y(s) + K_0(\alpha(t))y'(s))ds,$$
$$= K_1(\alpha(t))\,{}_0^{RL}I_t^{1-\alpha(t)}y(t) + K_0(\alpha(t))\,{}_0^{C}D_t^{\alpha(t)}y(t),$$
$$= K_1(\alpha(t))\,{}_0^{RL}D_t^{\alpha(t)-1}y(t) + K_0(\alpha(t))\,{}_0^{C}D_t^{\alpha(t)}y(t), \quad (38)$$

and, $K_1(\alpha(t))$, $K_0(\alpha(t))$ are rely on only $\alpha(t)$. By using NFDM and GL-approximation, it is possible to discretize (38) as:

$$
\begin{aligned}
&{}_0^{CPC}D_t^{\alpha(t)}y(t)|_{t=t^n} = \frac{K_1(\alpha(t))}{(\Theta(\Delta t))^{\alpha(t)-1}}(y_{n+1} + \textstyle\sum_{i=1}^{n+1}\omega_i y_{n+1-i}) \\
&+\frac{K_0(\alpha(t))}{(\Theta(\Delta t))^{\alpha(t)}}(y_{n+1} - \textstyle\sum_{i=1}^{n+1}\mu_i y_{n+1-i} - q_{n+1}y_0), \\
&= \rho_0 y(t_n) + \rho_1 y(t_n - \Delta t),
\end{aligned} \tag{39}
$$

and, $\omega_0 = 1$, $\omega_i = (1 - \frac{\alpha(t)}{i})\omega_{i-1}$, $t^n = n(\Theta(\Delta t)), \Delta t = \frac{T_f}{N_n}$, N_n is a natural number. $\mu_i = (-1)^{i-1}\binom{\alpha(t)}{i}$, $\mu_1 = \alpha(t)$, $q_i = \frac{i^{\alpha(t)}}{\Gamma(1-\alpha(t))}$ and $i = 1,2,\ldots,n+1$. Additionally, we assume that [22]:

$$
0 < \mu_{i+1} < \mu_i < \ldots < \mu_1 = \alpha(t) < 1,
$$
$$
0 < q_{i+1} < q_i < \ldots < q_1 = \frac{1}{\Gamma(-\alpha(t)+1)} .
$$

Remark 1 *We have the Caputo operator-based discretization of the finite difference approach (C-NFDM), if* $K_0(\alpha) = 1$ *and* $K_1(\alpha) = 0$ *in (39).*

5.2 Stability of CPC-GLNFDM

Evaluating the stability of the suggested approach (39) needs examining a validation issue for a fractional order linear delay differential equation model [17]. Let's examine this equation's approximative solution, $y(t_n) = y_n$. Afterwards, equation (37) is rewritten as follows by utilizing (CPC-NFDM) with the relation (10).

$$
\begin{aligned}
&\frac{K_1(\alpha(t))}{\Theta(\Delta t)^{\alpha(t)-1}}(y_{n+1} + \sum_{i=1}^{n+1}\omega_i y_{n+1-i}) + \frac{K_0(\alpha(t))}{\Theta(\Delta t)^{\alpha(t)}}(y_{n+1} - \sum_{i=1}^{n+1}\mu_i y_{n+1-i} - q_{n+1}y_0) \\
&\qquad = (\rho_0 y(t_n) + \rho_1 y(t_{n-q})),
\end{aligned} \tag{40}
$$

we put

$$
gg = \frac{K_1(\alpha(t))}{\Theta(\Delta t)^{\alpha(t)-1}} ,
$$

and

$$
gg1 = \frac{K_0(\alpha(t))}{\Theta(\Delta t)^{\alpha(t)}} ,
$$

we get:

$$
\begin{aligned}
&y_{n+1} = \frac{1}{(gg+gg1)\rho_0}(gg1\textstyle\sum_{i=1}^{n+1}\mu_i y_{n+1-i} - q_{n+1}y_0 - gg\textstyle\sum_{i=1}^{n+1}\omega_i y_{n+1-i} \\
&-(\rho_0 y(t_n) + \rho_1 y(t_{n-q}))).
\end{aligned} \tag{41}
$$

We have

$$
\frac{1}{(gg+gg1)} < 1,
$$

$$
y_1 \le y_0,
$$

$$
y_n \le y_{n-1} \le y_{n-2} \le \ldots \le y_1 \le y_0.
$$

Thus, the suggested strategy is appropriate and stable.

6. Numerical Experiments

In order to solve The fractional optimization systems with variable order hybrid FracInt (30) with the transversality criteria (31), we used the CPC-NFDM and Grünwald-Letnikov approximation in this part. We have selected certain parameter values from the literature for the model's real data fitting (12), and Egypt's data collection is used to fit the remaining values [27]. Different parameter values are obtained, as Table (1) illustrates. Only the term size balance in the equations is achieved by the positive parameters B_1, B_2. When using the parameter values indicated in Table (1), the following beginning conditions are:

$E(0) = 0$, $H(0) = 3$, $I(0) = 2$, $A(0) = 5$, $R(0) = 0$, $D(0) = 0$, and $S(0) = (\frac{100500159}{27193}) - 10$

Table 2 : demonstrates the objective functional values for both the classical and FracInt cases with $\tau = 0.4, T_f = 200$, and using different $\alpha, \alpha(t)$. It is obvious that Compared to the classical one, the variable-order FracInt objective functional values perform better.

Table (2): Comparing the values of $\tau = 0.4, T_f = 200$, using different $\alpha, \alpha(t)$.

$\alpha(t)$	J	α	J
$0 < t \leq 10$ $\alpha_0 = 1$ $10 < t \leq 200$ $\alpha_1 = 0.99$	53.5935	0.99	3.0559×10^4
$0 < t \leq 10$ $\alpha_0 = 1$ $10 < t \leq 200$ $\alpha_1 = 0.88$	5.8567	0.88	3.1474×10^4
$0 < t \leq 10$ $\alpha_0 = 1$ $10 < t \leq 200$ $\alpha_1 = 0.7$	8.2805	0.7	3.2384×10^4

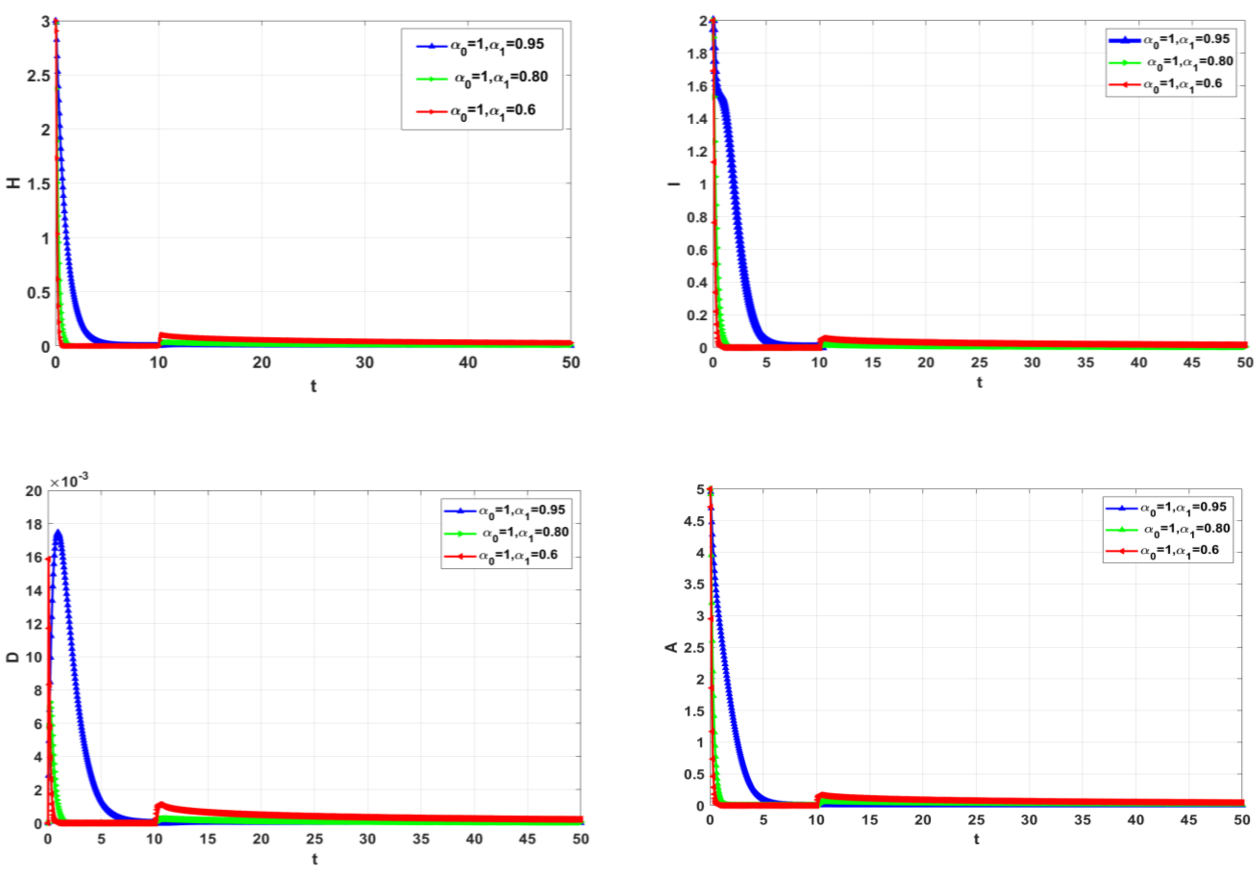

Figure 1: Simulation for optimality system at $\tau = 0.4$, and different $\alpha(t)$.

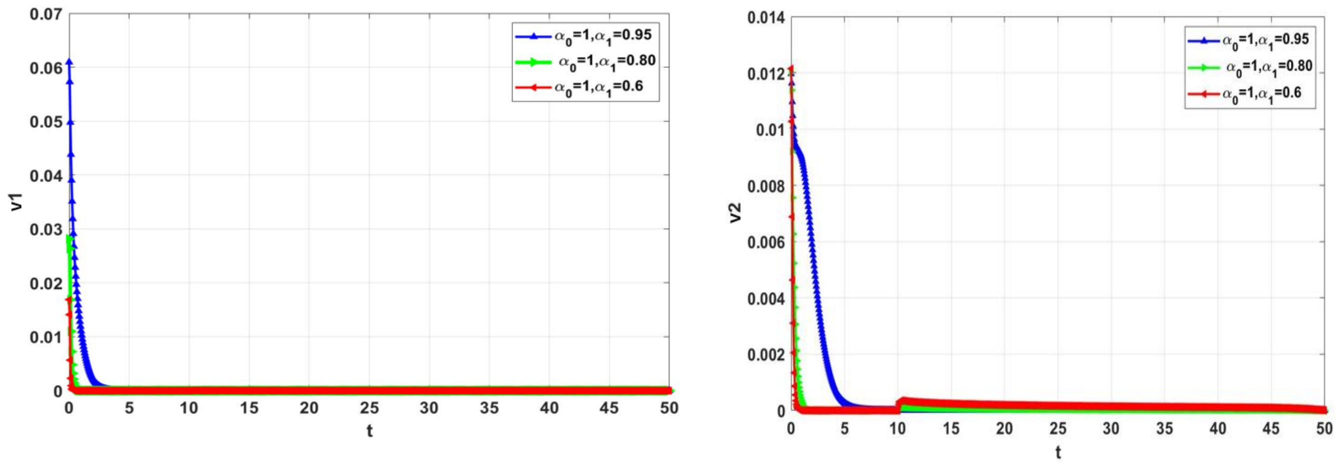

Figure 2: Simulation for control variable at $\tau = 0.4$, and different $\alpha(t)$.

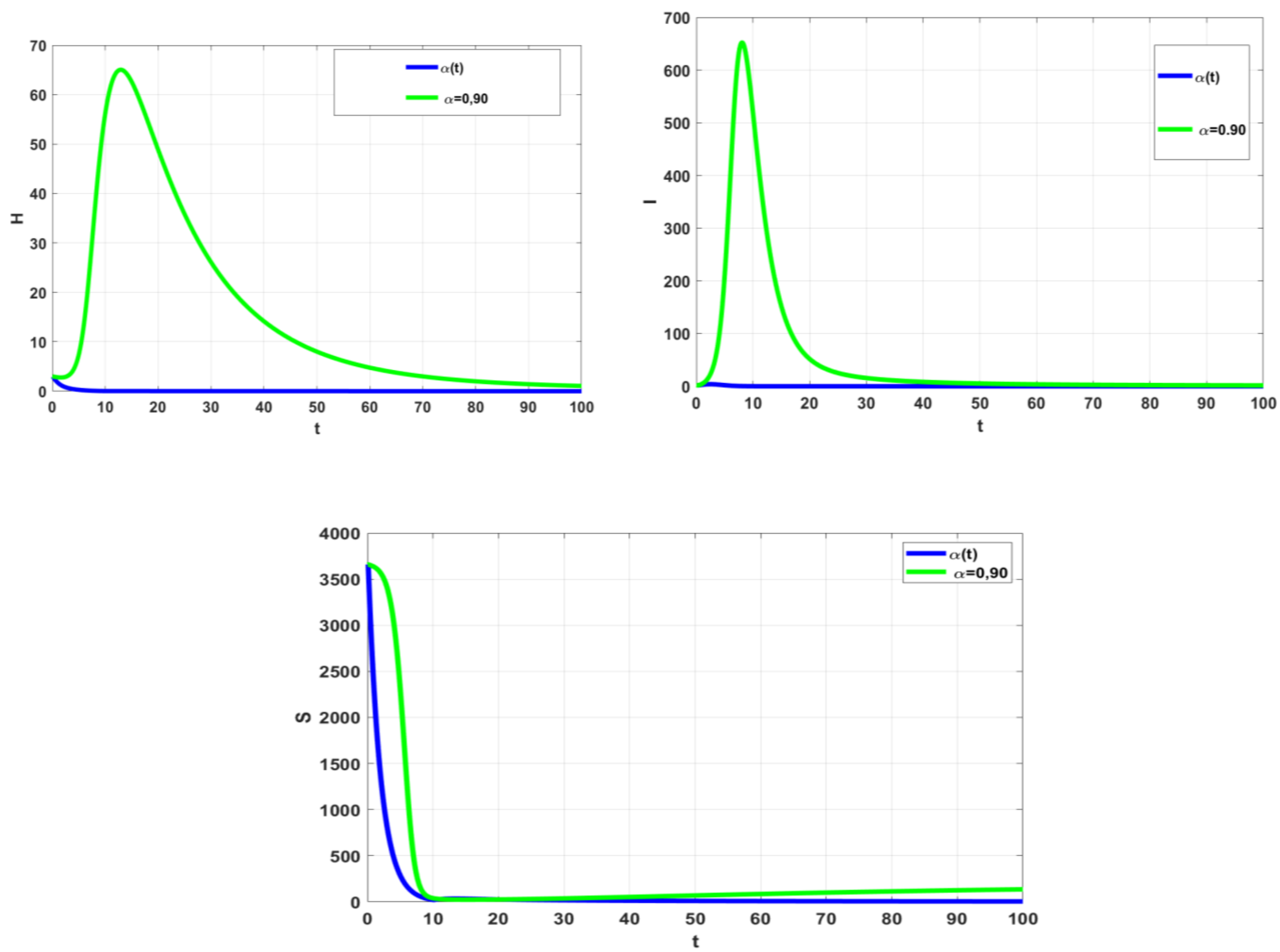

Figure 3: Simulation for optimality system at $\tau = 0.3$, $\alpha(t) = \alpha_0 = 1,\ if\ \ 0 < t \leq 10,\ \ \alpha_1 = 0.90\ \ if\ \ 10 < t \leq 100$

7. Conclusions

Studying the best time-delayed control schemes for the Covid-19 hybrid variable-order FracInt mathematical model is the main aim of this work. The variable-order derivative, which is better than the Caputo fractional operator, is determined by applying the variable-order Caputo derivative and the variable-order integral of Riemann-Liouville. The suggested model's stability study was presented. Boundedness, positivity, existence, and, Basic Reproductive Number are discussed and demonstrated in this model. We have effectively implemented a type of Pontryagin's maximum concept to minimize the spread of illness among healthy individuals. We introduced (CPC-GLNFDM) to analyze the optimality system and numerically derived the optimality requirements. Finally, According to the numerical results, the fractional optimal control system is only more successful for part of the time period. Consequently, we suggest a system in which the derivative order changes over time, becoming either fractional or integer when it is more beneficial for infection disease. It turns out that this variable-order fractional model, which is known as FractInt, works effectively for controlling the infectious disease. In the future, we will use the FracInt variable-order operator for solving numerically epidemic model diseases such as lumpy-skin and monkey-pox diseases.

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