Super Edge Magic Harmonious labeling for Certain Graphs

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ARTICLE INFO

Article history:
Received 17 November 2023
Received in revised form 3 December 2023
Accepted 4 December 2023
Available online 7 December 2023

Keywords
Super bimagic labeling;
harmonious trimagic labeling;
The sunflower graph;
The splitting graph.
2020 Mathematics Subject Classification: 05C78, 05C90

ABSTRACT

Edge labelling of graphs has received a lot of attention in the last few years. Both graph theory, networks, and discrete mathematics are fields that are still interested in this area. It is yet uncertain for many graphs whether super edge magic harmonious labeling exists or not. A graph $\Gamma=(V(\Gamma),E(\Gamma))$ with $p=|V(\Gamma)|$ vertices and $q=|E(\Gamma)|$ edges, is called an edge bimagic harmonious graph if there exists a bijective mapping $\Psi: [V(\Gamma) \cup E(\Gamma)] \rightarrow \{1,2,3,\ldots,p+q\}$ such that for each edge $xy \in E(\Gamma)$, the value of the formula $[(\Psi(x)+\Psi(y)) \mod(q)+\Psi(xy)]=K_1$ or $K_2$, where $K_1$ is a constant. If there exist three constants $K_1$, $K_2$, and $K_3$, it is said to be edge trimagic harmonious graph. We demonstrate in this study that the wheel graph $W_n$, and the splitting graph of odd cycle are super edge bimagic harmonious graphs. Furthermore, we point out that the sunflower graph and the double sunflower graph are super edge trimagic harmonious graphs.

1. Introduction

Everyday issues can be expressed visually through the use of graphs in numerous situations. Graph theory is relevant to all other fields, whether they are applied or pure. Labelling is one of the many fascinating issues covered in graph theory. A mapping of integers to vertices or edges in a graph according to predetermined standards is called a labeling of the graph. There are many kinds of labeling, such as graceful labeling, total edge irregularity labeling, harmonious labeling, total edge irregularity reflexive labeling, magic labeling, anti-magic labeling, Zumkeller and half Zumkeller labeling, geometric labeling, mean labeling and irregular labeling, etc.


Graph labelling techniques are utilized for communication network addressing system problems, fast communication in sensor networks, fault-tolerant system design using facility graphs, coding theory problems involving the creation of effective radar type codes, and mobile ad hoc network problems. [8,9].

See an up-to-date survey of graph labelling [10] for an in-depth overview of the subject of conflict.

In this paper, all graphs will be finite, undirected, and simple connected graphs, for a graph $\Gamma=(V,E)$, let $P=|V(\Gamma)|$ be the cardinality of vertices $V(\Gamma)$ and $q=|E(\Gamma)|$ be that of $E(\Gamma)$. We investigate and improve the concept of super edge bimagic and trimagic harmonious labeling of graphs. We prove that the wheel graph, and the splitting graph of cycle are super edge bimagic harmonious graphs. Furthermore, the sunflower, the double sunflower are super edge trimagic harmonious graphs.

Definition 1.1. [11] (1) A graph $\Gamma=(V(\Gamma),E(\Gamma))$ with $p=|V(\Gamma)|$ vertices and $q=|E(\Gamma)|$ edges is said to be an edge magic harmonious graph if there exists a bijective mapping $\Psi: [V(\Gamma) \cup E(\Gamma)] \rightarrow \{1,2,3,\ldots,p+q\}$ such that for each edge $xy \in E(\Gamma)$, the value of the formula $[(\Psi(x)+\Psi(y)) \mod(q)+\Psi(xy)]=K_1$ where $K_1$ is a constant.

(2) A graph $\Gamma=(V(\Gamma),E(\Gamma))$ with $p=|V(\Gamma)|$ vertices and $q=|E(\Gamma)|$ edges is said to be an edge bimagic harmonious graph if there exists a bijective mapping $\Psi: [V(\Gamma) \cup E(\Gamma)] \rightarrow \{1,2,3,\ldots,p+q\}$ such that for each edge $xy \in E(\Gamma)$, the value of the formula $[(\Psi(x)+\Psi(y)) \mod(q)+\Psi(xy)]=K_1$ or $K_2$, where $K_1$ and $K_2$ are constant.
A graph $\Gamma = (V(\Gamma), E(\Gamma))$ with $p = |V(\Gamma)|$ vertices and $q = |E(\Gamma)|$ edges is said to be an edge trimagic harmonious graph if there exists a bijective mapping $\Psi : [V(\Gamma) \cup E(\Gamma)] \rightarrow \{1, 2, 3, \ldots, p + q\}$ such that for each edge $xy \in E(\Gamma)$, the value of the formula $\left[\left(\Psi(x) + \Psi(y)\right) \mod (q) + \Psi(xy)\right]$ is $K_1$ or $K_2$ or $K_3$, where $K_1, K_2,$ and $K_3$ are constant.

**Definition 1.2.** [11]. If the graph $\Gamma$ has the extra property that the vertex labels are 1 to $p = |V(\Gamma)|$, then an edge bimagic harmonious labeling of the graph $\Gamma = (V(\Gamma), E(\Gamma))$ becomes a super edge bimagic harmonious labeling.

## 2. Super edge bimagic harmonious labeling of the wheel graph

**Theorem 2.1.** For every positive integer $n \geq 3$, the wheel graph $W_n$ is super edge bimagic harmonious graph with bimagic harmonious numbers $k_1 = 3n + 1$ and $k_2 = 4n$ for $n$ is even, while $k_2 = 4n + 1$ for $n$ is odd.

**Proof.** Let $\{x_0, x_1, \ldots, x_n\}$ be the vertices of the rim of wheel with hub vertex $x_0$, the edges of $W_n$ will be $\{x_0 x_i, x_i x_{i+1}, i \in [1, n-1]\} \cup \{x_n x_1\}$ So, $p = |V(W_n)| = n + 1$ and $q = |E(W_n)| = 2n$.

We define the labeling function $\Psi_1 : [(V \cup E(W_n))] \rightarrow \{1, 2, 3, \ldots, 3n + 1\}$ as follows:

$$
\Psi_1(x_0) = n + 1,
\Psi_1(x_i) = i, \quad i \in [1, n],
\Psi_1(x_0 x_i) = \begin{cases} 
2n - i, & \text{if } i \in [1, n - 2]; \\
3n + 1, & \text{if } i = n - 1; \\
3n, & \text{if } i = n.
\end{cases}
$$

For $n$ even,

$$
\Psi_1(x_0 x_{i+1}) = \begin{cases} 
3n - 2i, & \text{for } i \in [1, \frac{n}{2}]; \\
4n - 2i - 1, & \text{for } i \in \left[\frac{n}{2} + 1, n - 1\right]; \\
3n - 1, & \text{for } i = n.
\end{cases}
$$

For $n$ odd,

$$
\Psi_1(x_i x_{i+1}) = \begin{cases} 
3n - 2i, & \text{for } i \in [1, \frac{n - 1}{2}]; \\
4n - 2i, & \text{for } i \in \left[\frac{n + 1}{2}, n - 1\right]; \\
2n, & \text{for } i = n.
\end{cases}
$$

To demonstrate that $\Psi_1$ is a super edge bimagic harmonious labeling:

For the edges $x_0 x_i$, $i \in [1, n - 2]$

$$
\left[\left(\Psi_1(x_0) + \Psi_1(x_i)\right) \mod (q) + \Psi_1(x_0 x_i)\right] = \left[(n + 1 + i) \mod (2n) + 2n - i\right] 3n + 1 = K_1.
$$

For the edges $x_0 x_{n-1}$

$$
\left[\left(\Psi_1(x_0) + \Psi_1(x_{n-1})\right) \mod (q) + \Psi_1(x_0 x_{n-1})\right] = \left[(n + 1 + n - 1) \mod (2n) + 3n + 1\right] = 3n + 1 = K_1.
$$
For the edges $x_0x_n$
\[ [(\Psi_1(x_0) + \Psi_1(x_n)) \mod(q) + \Psi_1(x_0x_n)] = [(n + 1 + n) \mod(2n) + 3n] = 3n + 1 = K_1. \]

For the edges $x_ix_{i+1}$, $i \in [1, \frac{n}{2}]$ and $n$ is even, or $i \in [1, \frac{n-1}{2}]$ and $n$ is odd
\[ [(\Psi_1(x_i) + \Psi_1(x_{i+1})) \mod(q) + \Psi_1(x_ix_{i+1})] = [(i + i + 1) \mod(2n) + 3n - 2i] = 3n + 1 = K_1. \]

For the edges $x_ix_{i+1}$, $i \in [\frac{n}{2} + 1, n - 1]$ and $n$ is even
\[ [(\Psi_1(x_i) + \Psi_1(x_{i+1})) \mod(q) + \Psi_1(x_ix_{i+1})] = [(i + i + 1) \mod(2n) + 4n - 2i - 1] = 4n = K_2. \]

For the edges $x_ix_{i+1}$, $i \in [\frac{n+1}{2}, n - 1]$ and $n$ is odd
\[ [(\Psi_1(x_i) + \Psi_1(x_{i+1})) \mod(q) + \Psi_1(x_ix_{i+1})] = [(i + i + 1) \mod(2n) + 4n - 2i] = 4n + 1 = K_2. \]

For the edges $x_ix_n$, $n$ is even
\[ [(\Psi_1(x_i) + \Psi_1(x_n)) \mod(q) + \Psi_1(x_ix_n)] = [(n + 1) \mod(2n) + 3n - 1] = 4n = K_2. \]

For the edges $x_ix_n$, $n$ is odd
\[ [(\Psi_1(x_i) + \Psi_1(x_n)) \mod(q) + \Psi_1(x_ix_n)] = [(n + 1) \mod(2n) + 2n] = 3n + 1 = K_1. \]

Clearly, for each edge $xy \in E(W_n)$, the value of $[(\Psi_1(x) + \Psi_1(y)) \mod(q) + \Psi_1(xy)]$ provides any one of the bimagic constants $k_1 = 3n + 1$ and $k_2 = 4n$ when $n$ is even, while $k_2 = 4n + 1$ when $n$ is odd. Therefore, a super edge bimagic harmonious labelling for all $n$ is allowed by the wheel graphs $W_n$.

**Example 2.2.** In Fig. 1 we present $W_{13}$ with super edge bimagic harmonious labeling with bimagic harmonious numbers $k_1 = 40$ and $k_2 = 53$, and $W_{14}$ with super edge bimagic labeling with bimagic harmonious numbers $k_1 = 43$ and $k_2 = 56$, respectively.

![Figure 1](image)

3. **Super edge bimagic harmonious labeling of the splitting graph $S^-$ ($C_n$)**

**Theorem 3.1.** For any positive odd integer $n$, the splitting graph $S^-$ ($C_n$), $n \geq 3$ is super edge bimagic harmonious labeling graph with bimagic harmonious numbers $k_1 = 5n$ and $k_2 = 6n$. 
Proof. The splitting graph $S'(C_n), n \geq 3$ has vertex set $V[S'(C_n)] = \{ x_i, y_i, i \in [1, n] \}$ and its edges are $E[S'(C_n)] = \{ x_i x_{i+1}, y_i y_{i+1}, y_i x_i, y_n x_n, x_n y_1 \}$. In the graph $S'(C_n)$, we have $p = 2n$ vertices and $q = 3n$ edges.

When $n \geq 3$, is odd integer, we define the labeling function $\Psi_2 : [V \cup E(S'(C_n))] \to \{1, 2, 3, ..., 5n\}$ as follows:

$$\Psi_2(x_i) = n + i, \quad i \in [1, n],$$

$$\Psi_2(y_i) = i, \quad i \in [1, n],$$

$$\Psi_2(x_i y_{i+1}) = \begin{cases} 3n - 2i - 1, & \text{if } i \in \left[1, \frac{n - 3}{2}\right]; \\ 6n - 2i - 1, & \text{if } i \in \left[\frac{n - 1}{2}, n - 1\right]; \\ 5n - 1, & \text{if } i = n; \end{cases}$$

$$\Psi_2(x_i y_{i+1}) = \begin{cases} 5n - 2i - 1, & \text{if } i \in [1, n - 1]; \\ 4n - 1, & \text{if } i = n; \end{cases}$$

$$\Psi_2(y_i x_{i+1}) = \begin{cases} 4n - 2i - 1, & \text{if } i \in [1, n - 1]; \\ 3n - 1, & \text{if } i = n. \end{cases}$$

To demonstrate that $\Psi_2$ is a super edge bimagic harmonious labeling:

For the edges $x_i x_{i+1}$, $1 \leq i \leq \frac{n - 3}{2}$,

$$[\Psi_2(x_i) + \Psi_2(x_{i+1})] \mod q + \Psi_2(x_i x_{i+1}) = [(2n + 2i + 1) \mod (3n) + 3n - 2i - 1] = 5n = K_1.$$  

For the edges $x_i x_{i+1}$, $\frac{n - 1}{2} \leq i \leq n - 1$,

$$[\Psi_2(x_i) + \Psi_2(x_{i+1})] \mod q + \Psi_2(x_i x_{i+1}) = [(2n + 2i + 1) \mod (3n) + 6n - 2i - 1] = 5n = K_1.$$  

For the edges $y_i x_n$,

$$\left[\left(\Psi_2(y_n) + \Psi_2(x_1)\right) \mod q + \Psi_2(x_n x_1)\right] = [(3n + 1) \mod (3n) + 5n - 1] = 5n = K_1.$$  

For the edges $y_i x_{i+1}$, $1 \leq i \leq n - 1$,

$$\left(\Psi_2(x_i) + \Psi_2(x_{i+1})\right) \mod q + \Psi_2(y_i x_{i+1}) = [(2n + 1) \mod (3n) + 5n - 2i - 1] = 5n = K_1.$$  

For the edges $y_n x_1$,

$$\left(\Psi_2(y_n) + \Psi_2(x_1)\right) \mod q + \Psi_2(y_n x_1) = [(2n + 1) \mod (3n) + 3n - 1] = 5n = K_1.$$  

For the edges $x_i y_{i+1}$, $1 \leq i \leq n$,

$$\left(\Psi_2(x_i) + \Psi_2(y_{i+1})\right) \mod q + \Psi_2(x_i y_{i+1}) = [(n + 2i + 1) \mod (3n) + 5n - 2i - 1] = 6n = K_2.$$  

For the edges $y_1 y_n$,

$$\left(\Psi_2(y_1) + \Psi_2(x_n)\right) \mod q + \Psi_2(y_1 y_n) = [(2n + 1) \mod (3n) + 4n - 1] = 6n = K_2.$$
Obviously, the edge labels are all distinct and for each edge $xy \in E(S'(C_9))$, the value of the formula $[(\Psi_2(x) + \Psi_2(y)) \mod(q) + \Psi_2(xy)]$ provides either of the magic constants $k_1 = 5n$ or $k_2 = 6n$ when $n$ is odd. Therefore, a super edge bimagic harmonious labelling is allowed by the splitting graph $S'(C_n)$.

**Example 3.2.** In Fig. 2, we present the splitting graph $S'(C_9)$ with an edge bimagic harmonious labelling and bimagic constants $k_1 = 55$, and $k_2 = 64$.

![Figure 2: (a) The splitting graph $S'(C_9)$ with bimagic harmonious constants](image)

4. Super edge trimagic harmonious labeling of the graph $SF_n$

**Definition 4.1.** For a vertex $x \in V(\Gamma')$, the open neighborhood set $N(x)$ is the set of vertices which are adjacent to $x$ in $\Gamma$. Duplication of an edge $e = x_ix_{i+1}$ in a graph $\Gamma$ by a new vertex $y_i$ produces a new graph $\Gamma'$ such that $N(y_i) = \{x_i, x_{i+1}\}$. The sunflower graph $SF_n$, is defined as a graph obtained by starting with an $n$-cycle $C_n = \{x_1, x_2, ..., x_n\}$ and duplicating every edge by new vertices $\{y_1, y_2, ..., y_n\}$ with $y_i$ connected to $x_i$ and $x_{i+1}$, so

$$E(SF_n) = \{x_ix_i, y_iy_{i+1}, x_ix_{i+1}, \quad i \in [1, n]\} \cup \{y_nx_1, x_nx_1\}.$$  

**Theorem 4.2.**

(i) The sunflower graph $SF_n$ is super edge trimagic harmonious graph with trimagic harmonious numbers $k_1 = 5n, k_2 = 7n$ and $k_3 = 6n$ when $n \geq 3$ is odd integer.

(ii) The sunflower graph $SF_n$ is super edge trimagic harmonious graph with trimagic harmonious numbers $k_1 = 5n, k_2 = 7n$ and $k_3 = 6n - 1$ when $n$ is even integer.

**Proof.** The sunflower graph $SF_n$ has $p = 2n$ vertices and $q = 3n$ edges. 

Case (1): When $n \geq 3$ is odd integer, define the labeling function

$$\Psi_3 : (V \cup E)(SF_n) \rightarrow \{1, 2, 3, ..., 5n\}$$  
as follows:

$$\Psi_3(x_i) = i, \quad i \in [1, n],$$
\[ \Psi_3(y_i) = n + i, \quad i \in [1, n], \]
\[ \Psi_3(y_{i+1}) = \begin{cases} 4n - 2i - 1, & \text{if } i \in [1, n-1]; \\ 5n - 1, & \text{if } i = n; \end{cases} \]
\[ \Psi_3(x_i x_{i+1}) = \begin{cases} 5n - 2i - 1, & \text{if } i \in [1, n-1]; \\ 4n - 1, & \text{if } i = n; \end{cases} \]
\[ \Psi_3(x_i y_i) = \begin{cases} 5n - 2i, & \text{if } i \in \left[1, \frac{n-1}{2}\right]; \\ 4n - 2i, & \text{if } i \in \left[\frac{n+1}{2}, n-1\right]; \\ 5n, & \text{if } i = n. \end{cases} \]

To demonstrate that \( \Psi_3 \) is a super edge trimagic harmonious labeling:

For the edges \( y_i x_{i+1}, \quad i \in [1, n - 1], \)
\[ \left(\Psi_3(y_i) + \Psi_3(x_{i+1})\right) \mod (q) + \Psi_3(y_{i+1} x_{i+1}) = [(n + i + i + 1) \mod (3n) + 4n - 2i - 1] = 5n = K_1. \]

For the edges \( y_n x_1, \)
\[ \left(\Psi_3(y_n) + \Psi_3(x_1)\right) \mod (q) + \Psi_3(y_n x_1) = [(2n + 1) \mod (3n) + 5n - 1] = 7n = K_2. \]

For the edges \( x_i x_{i+1}, \quad i \in [1, n - 1] \)
\[ \left(\Psi_3(x_i) + \Psi_3(x_{i+1})\right) \mod (q) + \Psi_3(x_i x_{i+1}) = [(i + i + 1) \mod (3n) + 5n - 2i - 1] = 5n = K_1. \]

For the edges \( x_1 x_n \)
\[ \left(\Psi_3(x_1) + \Psi_3(x_n)\right) \mod (q) + \Psi_3(x_1 x_n) = [(1 + n) \mod (3n) + 4n - 1] = 5n = K_1. \]

For the edges \( x_i y_i, \quad i \in [1, \frac{n-1}{2}], \)
\[ \left(\Psi_3(x_i) + \Psi_3(y_i)\right) \mod (q) + \Psi_3(x_i y_i) = [(i + n + i) \mod (3n) + 5n - 2i] = 6n = K_3. \]

For the edges \( x_i y_i, \quad i \in [\frac{n+1}{2}, n - 1], \)
\[ \left(\Psi_3(x_i) + \Psi_3(y_i)\right) \mod (q) + \Psi_3(x_i y_i) = [(i + n + i) \mod (3n) + 4n - 2i] = 5n = K_1. \]

For the edge \( x_n y_n \)
\[ \left(\Psi_3(x_i) + \Psi_3(y_i)\right) \mod (q) + \Psi_3(x_i y_i) = [(n + 2n) \mod (3n) + 5n] = 5n = K_1. \]
Case (2): When \( n \geq 4 \) is even integer. The labeling function \( \Psi_3 \) defined as in case (1) but with some modifications which are given by

\[
\Psi_3(x_i, x_{i+1}) = \begin{cases} 
5n - 2i - 1, & \text{if } i \in \left[1, \frac{n}{2}\right]; \\
6n - 2i - 2, & \text{if } i \in \left[\frac{n}{2} + 1, n - 1\right]; \\
5n - 2 & \text{if } i = n.
\end{cases}
\]

\[
\Psi_3(x_i, y_i) = \begin{cases} 
4n - 2i, & \text{if } i \in [1, n-1]; \\
5n, & \text{if } i = n.
\end{cases}
\]

To demonstrate that \( \Psi_3 \) is a super edge trimagic harmonious labeling:

For the edges \( x_i, x_{i+1} \), \( i \in \left[1, \frac{n}{2}\right] \),

\[
\left[\left(\Psi_3(x_i) + \Psi_3(x_{i+1})\right) \mod (q) + \Psi_3(x_i, x_{i+1})\right] = \left[(i + i + 1) \mod (3n) + 5n - 2i - 1\right] = 5n = K_1.
\]

For the edges \( x_i, x_{i+1} \), \( i \in \left[\frac{n}{2} + 1, n - 1\right] \),

\[
\left[\left(\Psi_3(x_i) + \Psi_3(x_{i+1})\right) \mod (q) + \Psi_3(x_i, x_{i+1})\right] = \left[(i + i + 1) \mod (3n) + 6n - 2i - 2\right] = 6n - 1 = K_3.
\]

For the edges \( x_1, x_n \),

\[
\left[\left(\Psi_3(x_1) + \Psi_3(x_n)\right) \mod (q) + \Psi_3(x_1, x_n)\right] = \left[(1 + n) \mod (3n) + 5n - 2\right] = 6n - 1 = K_3.
\]

For the edges \( x_i, y_i \), \( i \in [1, n - 1] \),

\[
\left[\left(\Psi_3(x_i) + \Psi_3(y_i)\right) \mod (q) + \Psi_3(x_i, y_i)\right] = \left[(i + n + i) \mod (3n) + 4n - 2i\right] = 5n = K_1.
\]

For the edge \( x_n, y_n \),

\[
\left[\left(\Psi_3(x_n) + \Psi_3(y_n)\right) \mod (q) + \Psi_3(x_n, y_n)\right] = \left[(n + 2n) \mod (3n) + 5n\right] = 5n = K_1.
\]

Therefore, an edge trimagic harmonious labelling for all \( n \) is allowed by the graph \( SF_n \).

**Example 4.3.** In Fig. 3, we present \( SF_9 \) with an edge trimagic harmonious labelling and trimagic constants \( k_1 = 45 \), \( k_2 = 63 \) and \( k_3 = 54 \). and \( SF_{10} \) with the trimagic harmonious constants are \( k_1 = 50 \), \( k_2 = 70 \) and \( k_3 = 59 \).

**Figure 3:** (a) \( SF_9 \) with \( k_1 = 45 \), \( k_2 = 63 \) and \( k_3 = 54 \) (b) \( SF_{10} \) with \( k_1 = 50 \), \( k_2 = 70 \) and \( k_3 = 59 \)
5. Super edge trimagic harmonious labeling of the double sunflower graph

**Definition 5.1.** A double sunflower graph of order \( n \), represented by \( DSF_n \), is a graph that is obtained from the sunflower graph \( SF_n \) by inserting a new vertex \( u_i \) on each edge \( x_i x_{i+1} \) on the rim of the cycle and adding edges \( y_i u_i \) for each \( i \in [1,n] \).

**Theorem 5.2.** For every integer \( n \geq 3 \), the double sunflower graph \( DSF_n \) is super edge trimagic harmonious graph with trimagic numbers \( k_1 = 8n \), \( k_2 = 10n - 1 \), and \( k_3 = 11n \).

**Proof.** The double sunflower graph \( DSF_n \) has vertex set \( \{ x_i, u_i, y_i, i \in [1,n] \} \) and edge set \( E[DSF_n] = \{ x_i u_i, x_i y_i, u_i y_i, i \in [1,n] \} \cup \{ u_i x_{i+1}, y_i x_{i+1}, i \in [1,n-1] \} \cup \{ u_n x_1, y_n x_1 \} \). So, \( p = 3n \) vertices and \( q = 5n \) edges.

Define the labeling function \( \Psi_4 : \mathbb{V} \cup E(DSF_n) \rightarrow \{1, 2, 3, ..., 8n\} \) as follows:

- \( \Psi_4(x_i) = 2i - 1 \), \( i \in [1,n] \),
- \( \Psi_4(u_i) = 2i \), \( i \in [1,n] \),
- \( \Psi_4(y_i) = 2n + i \), \( i \in [1,n] \),
- \( \Psi_4(x_i y_i) = 6n - 3i + 1 \), \( i \in [1,n] \).

\[
\Psi_4(x_i u_i) = \begin{cases} 
8n - 4i + 1, & \text{for } i \in \left[1, \frac{n}{2}\right], n \text{ is even;} \\
10n - 4i, & \text{for } i \in \left[\frac{n}{2} + 1, n\right], n \text{ is even;} \\
8n - 2, & \text{for } i = n.
\end{cases}
\]

To demonstrate that \( \Psi_4 \) is a super edge trimagic harmonious labeling:

For the edges \( x_i y_i \), \( i \in [1,n] \)

\[
[(\Psi_4(x_i) + \Psi_4(y_i)) \mod(q) + \Psi_4(x_i y_i)] = [(2i - 1 + 2n + i) \mod(5n) + 6n - 3i + 1] = 8n = K_1.
\]

For the edges \( y_n x_1 \)

\[
[(\Psi_4(y) + \Psi_4(x_1)) \mod(q) + \Psi_4(y_n x_1)] = [(3n + 1) \mod(5n) + 8n - 1] = 11n = K_2.
\]
For the edges $x_iu_i, i \in \left[\frac{n}{2} + 1, n\right]$, if $n$ is even, and $i \in \left[\frac{n+3}{2}, n\right]$, if $n$ is odd,
\[
(\Psi_4(u_i) + \Psi_4(x_i)) \mod(0) + \Psi_4(xu_i) = [2i - 1 + 2i] \mod(5n) + 10n - 4i = 10n - 1 = K_3.
\]
It is clear that, for each edge $xy \in E(DSF_n)$, the value of $[(\Psi_4(x) + \Psi_4(y)) \mod(q) + \Psi_4(xy)]$ provides either of the trimagic constants $k_1 = 8n, k_2 = 10n - 1$, or $k_3 = 11n$. Therefore, a super edge trimagic harmonious labelling for all $n$ is allowed by the graph $DSF_n$.

**Example 5.3.** In Fig. 4, we present the double sunflower graph $DSF_n$ with bimagic harmonious constants $k_1 = 56, k_2 = 69$, and $k_3 = 77$ and $DSF_3$ with an edge bimagic harmonious labelling and bimagic constants $k_1 = 64, k_2 = 79$, and $k_3 = 88$.

![Figure 4](image1.png)

**Figure 4:** (a) $DSF_7$ with $k_1 = 56, k_2 = 69$ and $k_3 = 77$ (b) $DSF_8$ with $k_1 = 64, k_2 = 79$ and $k_3 = 88$

**Conclusion**

In the past few years, edge labeling of graphs has been studied heavily and this topics continue to be attractive in the field of graph theory and discrete mathematics. So far, many graphs are unknown if it is super edge harmonious or not. In this work, we prove that the wheel graph $W_n$ and the splitting graph of odd cycle are super edge bimagic harmonious graphs. Furthermore, we prove that the sunflower graph and the double sunflower graph are super edge trimagic harmonious graphs. In the following tables we summaries our results:

<table>
<thead>
<tr>
<th>Graph</th>
<th>$p$</th>
<th>$q$</th>
<th>$k_1$</th>
<th>$k_2$</th>
<th>$k_3$</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wheel graph $W_n$</td>
<td>$n + 1$</td>
<td>$2n$</td>
<td>$3n + 1$</td>
<td>$4n$</td>
<td>---</td>
<td>$n$ even</td>
</tr>
<tr>
<td>Wheel graph $W_n$</td>
<td>$n + 1$</td>
<td>$2n$</td>
<td>$3n + 1$</td>
<td>$4n + 1$</td>
<td>---</td>
<td>$n$ odd</td>
</tr>
<tr>
<td>Splitting graph $S'(C_n)$</td>
<td>$2n$</td>
<td>$3n$</td>
<td>$5n$</td>
<td>$6n$</td>
<td>---</td>
<td>$n$ odd</td>
</tr>
<tr>
<td>Sunflower graph $SF_n$</td>
<td>$2n$</td>
<td>$3n$</td>
<td>$5n$</td>
<td>$7n$</td>
<td>$6n$</td>
<td>$n$ odd</td>
</tr>
<tr>
<td>Sunflower graph $SF_n$</td>
<td>$2n$</td>
<td>$3n$</td>
<td>$5n$</td>
<td>$7n$</td>
<td>$6n - 1$</td>
<td>$n$ even</td>
</tr>
<tr>
<td>Double sunflower graph $DSF_n$</td>
<td>$3n$</td>
<td>$5n$</td>
<td>$8n$</td>
<td>$10n - 1$</td>
<td>$11n$</td>
<td></td>
</tr>
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References


